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Landau Levels on a Torus

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Landau levels have represented a very rich field of research, which has gained widespread attention after their application to the quantum Hall effect. In a particular gauge, the holomorphic gauge, they give a physical implementation of Bargmann's Hilbert space of entire functions. They have also been recognized as a natural bridge between Feynman's path integral and geometric quantization. We discuss here some mathematical subtleties involved in the formulation of the problem when one tries to study quantum mechanics on a finite strip of sides L_1, L_2 with a uniform magnetic field and periodic boundary conditions. There is an apparent paradox here: infinitesimal translations should be associated to canonical operators $[\mathfrak{p}_x, \mathfrak{p}_y] \propto i\hbar B$, and, at the same time, live in a Landau level of finite dimension $BL_1L_2/(hc/e)$, which is impossible from Wintner's theorem. The paper shows the way out of this conundrum.

1. INTRODUCTION

Landau levels were introduced in 1930 (see ref. 9). They found an important physical application only quite recently, after the discovery of the quantum Hall effect (see refs. 2, 4, and 5 and references therein). More recently, it has been recognized that the theory of Landau levels provides a general bridge between Feynman path integrals and "geometric quantization" in all cases where the classical phase space is equipped with a complex structure which makes it a Kaehler manifold [7]. From this general viewpoint, or to get a more realistic description of conducting thin films, it is important to understand the case of a finite region with suitable boundary conditions. If these correspond to a compact (smooth) manifold without boundary, the quantization condition of Kostant and Souriau (see ref. 11) and references therein) or, equivalently, Dirac's quantization condition for monopole charges requires that the total magnetic flux be quantized, i.e., it must be an integral

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multiple N of the universal constant hc/e. At the same time, the degeneracy of the ground state is finite and coincides with N, except for a topological correction (half the Euler characteristic of the manifold). A similar, approximate, result can be obtained by a semiclassical argument [9].

Consider now the simple case of a rectangular area with sides L_1, L_2 and periodic boundary conditions; the problem is formulated on a toroidal surface with a transverse magnetic field whose flux is BL_1L_2. Of course, this fact implies the presence of magnetic charges, hence Dirac's quantization. The problem is: What is the symmetry of the Hamiltonian? We expect that the classical symmetry of the torus ($S^1 \times S^1$) be realized as a projective representation, the two infinitesimal generators satisfying Heisenberg algebra with a central charge $\propto \hbar B$ (at least this is what happens in the noncompact R^2 case). But this is clearly incompatible (Wintner's theorem) with finite degeneracy of energy levels! While we cannot expect a spontaneous symmetry breaking in a system with a finite number of degrees of freedom, we know from geometric quantization that *not all* classical symmetries survive at the quantum level, only those which are lifted at the prequantum level and, second, respect the polarization (in Landau level language, those symmetries are preserved that leave the first Landau level invariant). The problem is: What exactly is happening on the torus?

To get an answer, we shall reconstruct the explicit form of the Landau levels in terms of sections of the Hermitian line bundle associated to the principal bundle with connection given by the magnetic potential **A**. The language of fiber bundles is the natural one to describe gauge fields and it is becoming more familiar to physicists especially after the advent of modern string theory. We shall explicitly construct the transition functions of the line bundle and find a natural orthonormal basis of holomorphic sections, which turn out to be Jacobi θ-functions. By inspection, it turns out that translation invariance is broken to a discrete subgroup $Z_N \times Z_N$, N being the monopole charge. This fact has the counterpart that the Hermitian operators which correspond to infinitesimal translations (in the noncompact case) *do not leave the Landau levels invariant*, i.e., *they do not commute with the Hamiltonian*: while formally commuting with the Hamiltonian as *differential operators*, they fail to respect the boundary conditions given by the bundle transition functions.

2. MAGNETIC FIELD ON THE TORUS

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the two-torus; we describe it in physical terms by identifying an atlas of four local charts specified as follows:

1. $\mathcal{U}_\alpha = \{0 < x < L_1,\ 0 < y < L_2\}$

2. $\mathcal{U}_\beta = \{\bar{x} < x < L_1 + \bar{x},\ 0 < y < L_2\}$
3. $\mathcal{U}_\gamma = \{0 < x < L_1,\ \bar{y} < y < L_2 + \bar{y}\}$
4. $\mathcal{U}_\delta = \{\bar{x} < x < L_1 + \bar{x},\ \bar{y} < y < L_2 + \bar{y}\}$

with some choice of constants $\bar{x}$ and $\bar{y}$. A uniform magnetic field B transverse to the surface $\mathbb{T}^2$ is represented by the translation-invariant two-form $\mathbf{B} = B\, dx \wedge dy$. The connection form $\mathbf{A}$, representing the magnetic potential, is defined in each local chart in such a way that $d\mathbf{A} = \mathbf{B}$. It is well known that a global one-form on $\mathbb{T}^2$ satisfying this condition does *not* exist, since otherwise $\int_{\mathbb{T}^2} \mathbf{B} = \int_{\mathbb{T}^2} d\mathbf{A} = 0$, by Stokes' theorem, while it holds that $\int_{\mathbb{T}^2} \mathbf{B} = BL_1L_2$. The problem is essentially the same as the presence of a Dirac string in the case of a three-dimensional magnetic monopole. For the sake of simplicity, we may define the local connection forms by the same formula $\mathbf{A} = \frac{1}{2}B(x\, dy - y\, dx)$ since the coordinates x, y are indeed differentiable within each local chart.[2] To characterize the connection form completely, we have to identify the *transition functions* which relate $\mathbf{A}_i$ to $\mathbf{A}_j$ for any pair (i, j) in the set $\{\alpha, \beta, \gamma, \delta\}$ and *for each connected component* of the overlap $\mathcal{U}_i \cap \mathcal{U}_j$; we have

$$\mathbf{A}_\beta(x, y) = \mathbf{A}_\alpha(x, y) \qquad (\bar{x} < x < L_1)$$

$$\mathbf{A}_\gamma(x, y) = \mathbf{A}_\alpha(x, y) \qquad (\bar{y} < y < L_2)$$

$$\mathbf{A}_\delta(x, y) = \mathbf{A}_\alpha(x, y) \qquad (\bar{x} < x < L_1, \bar{y} < y < L_2)$$

$$\mathbf{A}_\beta(x + L_1, y) = \mathbf{A}_\alpha(x, y) + d(\tfrac{1}{2}\, BL_1y + \varphi_{\alpha\beta}) \qquad (0 < x < \bar{x})$$

$$\mathbf{A}_\gamma(x, y + L_2) = \mathbf{A}_\alpha(x, y) + d(-\tfrac{1}{2}BL_2x + \varphi_{\alpha\gamma}) \qquad (0 < y < \bar{y})$$

$$\mathbf{A}_\delta(x + L_1, y + L_2) = \mathbf{A}_\alpha(x, y) + d(\tfrac{1}{2}B(L_1y - L_2x) + \varphi_{\alpha\delta})$$

$$(0 < x < \bar{x}, \quad 0 < y < \bar{y})$$

and similar transition functions for the other cases. The constants φ_{ij} are arbitrary at this level; they will play, however, a crucial role in the *lifting* to the associated line bundle which describes the quantum wave functions.[3]

3. THE HOLOMORPHIC GAUGE

We now make a gauge transformation to a special gauge which is particularly convenient in the quantization process. Let us introduce complex coordinates $z = x + iy$, $\bar{z} = x - iy$. The magnetic potential is given by

[2] The correct mathematical language to describe such a setup is that of algebraic geometry; a nice introduction for physicists can be round, for instance, in ref. 1 In this paper, we try to keep the mathematical jargon to a minimum.
[3] The constants $\varphi_{\alpha\beta}$ are connected to the fundamental cocycle $c_{\alpha\beta\gamma}$ of refs. 1 and 11.

$$\mathbf{A}(z, \bar{z}) = \frac{1}{2i} B\bar{z}\, dz - \frac{1}{4i} B\, d|z|^2 \tag{1}$$

which shows that by a gauge transformation, we can adopt a holomorphic form

$$\mathbf{A}^h = \frac{1}{2i} B\, \bar{z}\, dz$$

for which we have the transition functions

$$\begin{aligned}
\mathbf{A}^h_\beta(z) &= \mathbf{A}^h_\alpha(z) \qquad (\bar{x} < x < L_1) \\
\mathbf{A}^h_\gamma(z) &= \mathbf{A}^h_\alpha(z) \qquad (\bar{y} < y < L_2) \\
\mathbf{A}^h_\delta(z) &= \mathbf{A}^h_\alpha(z) \qquad (\bar{x} < x < L_1, \bar{y} < y < L_2) \\
\mathbf{A}^h_\beta(z + L_1) &= \mathbf{A}^h_\alpha(z) + d(-i\tfrac{1}{2}BL_1 z + \varphi_{\alpha\beta}) \qquad (0 < x < \bar{x}) \\
\mathbf{A}^h_\gamma(z + iL_2) &= \mathbf{A}^h_\alpha(z) + d(-\tfrac{1}{2}BL_2 z + \varphi_{\alpha\gamma}) \qquad (0 < y < \bar{y}) \\
\mathbf{A}^h_\delta(z + L_1\; iL_2) &= \mathbf{A}^h_\alpha(z) + d(-i\tfrac{1}{2}B(L_1 - iL_2)z + \varphi_{\alpha\delta}) \\
&\qquad (0 < x < \bar{x}, 0 < y < \bar{y})
\end{aligned}$$

with some new choice of constants φ_{ij}.

4. QUANTIZATION

The Hamiltonian for a charged particle is given by the minimal-coupling prescription. The local expression as a differential operator must be complemented by suitable boundary conditions which ensure self-adjointness. This is easily done in terms of a line bundle *associated* to **A** as defined in the previous section. The physical principle to adopt is the gauge principle, according to which

$$\left(i\hbar\partial_\mu - \frac{e}{c} A_\mu\right)\psi$$

is covariant under gauge transformations, in particular under the transition from one chart to another (here A_μ are the components of the gauge potential one-form $\mathbf{A} = \Sigma\, A_\mu\, dx^\mu$). This can be done directly in the holomorphic gauge, which is our choice for the sequel. As usual, the complex line bundle has transition functions obtained by exponentiating those which characterize $\mathbf{A}^h$:

$$\begin{aligned}
\psi_\beta(z) &= \psi_\alpha(z) \qquad (\bar{x} < x < L_1) \\
\psi_\gamma(z) &= \psi_\alpha(z) \qquad (\bar{y} < y < L_2)
\end{aligned}$$

$$\psi_\delta(z) = \psi_\alpha(z) \qquad (\bar{x} < x < L_1, \bar{y} < y < L_2)$$

$$\psi_\beta(z + L_1) = \psi_\alpha(z) \exp\left\{\frac{eBL_1}{2\hbar c} z + \phi_{\alpha\beta}\right\} \qquad (0 < x < \bar{x})$$

$$\psi_\gamma(z + iL_2) = \psi_\alpha(z) \exp\left\{-\frac{ieBL_2}{2\hbar c} z + \phi_{\alpha\gamma}\right\} \qquad (0 < y < \bar{y})$$

$$\psi_\delta(z + L_1 + iL_2) = \psi_\alpha(z) \exp\left\{\frac{eB(L_1 - iL_2)}{2\hbar c} z + \phi_{\alpha\delta}\right\}$$

$$(0 < x < \bar{x}, 0 < y < \bar{y})$$

where we have redefined the constants $\varphi \to \phi$ to absorb a common factor $ieB/\hbar c$. It is clear that both $\bar{\partial}\psi$ and $(\partial - z)\psi$ transform in the same way as ψ. We have to stress here that while ϕ_{ij} are totally arbitrary, they *must* be chosen once and for all to define the Hamiltonian; as we shall show, different choices correspond in general to unitarily equivalent, yet distinct, operators. The situation is rather different from the well-known Aharonov–Bohm case, where the various admissible boundary conditions yield inequivalent Hamiltonians.

The local expression of the Hamiltonian in terms of complex coordinates is easily found to be

$$H^h = -4\,\frac{\hbar^2}{2m}\left(\partial - \frac{eB}{2mc\hbar}\,\bar{z}\right)\bar{\partial} \tag{2}$$

($\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial\bar{z}$), where we dropped a zero-point energy term $\hbar\omega$.

We must now introduce the Hermitian structure which allows us to define the quantum inner product between wave functions. It is readily seen (e.g., starting from the Euclidean inner product in the real gauge and performing the gauge transformation to the holomorphic case) that the Hermitian structure is given by

$$h(\psi_1, \psi_2) = \exp\left\{-\frac{eB}{2\hbar c}|z|^2\right\}\overline{\psi_1}\psi_2$$

in terms of which we can define the quantum inner product

$$\langle\psi_1|\psi_2\rangle = \int_{\mathbb{T}^2} h(\psi_1, \psi_2)[dz]$$

where $[dz] \equiv (1/2i)\overline{dz} \wedge dz$. It is easy to check that there is a smooth match $h(\psi_i, \psi_i) = h(\psi_j, \psi_j)$ on each $\mathcal{U}_i \cap \mathcal{U}_j$ provided that $\Re(\phi_{ij})$ is suitably chosen.

To simplify the notation, let us introduce natural units adapted to the problem: let us use $(\hbar/m\omega)^{1/2}$ as length unit, where $\omega = eB/2mc$ is the Larmor frequency. Then we get the new transition functions which make $h(\psi, \phi)$ smooth. At this point, we can drop the chart index from the wave function: from our convention, there is an open set common to all local charts where the wave function is the same in all local charts and the transition functions merely represent the boundary conditions to be imposed on ψ.

$$\begin{aligned}\psi(z + L_1) &= \psi(z)\, \exp\{L_1 z + \tfrac{1}{2}L_1^2 + i\delta_1\}\\ \psi(z + iL_2) &= \psi(z)\, \exp\{-iL_2 z + \tfrac{1}{2}L_2^2 + i\delta_2\}\end{aligned} \tag{3}$$

It is also easily checked that these b.c. make the Hamiltonian Hermitian. [Hint: Use complex integration by parts in the form $\int_{\mathbb{T}^2} \overline{dz} \wedge dz\, \overline{\phi(z)}\, \bar{\partial}\psi(z) = \int_{\mathbb{T}^2} \overline{\phi(z)}\, d(\psi\, dz) = \oint \overline{\phi(z)}\psi(z)\, dz - \int_{\mathbb{T}^2} \overline{dz} \wedge dz\, \overline{\partial\phi(z)}\psi$.]

However, there is a consistency condition to be satisfied, which stems from a general theorem about Hermitian line bundles due to Weil [12] (a simple proof taken from ref. 11 is reproduced in Appendix B). In our case, it can be found as follows: by successively applying the previous relations, we get

$$\begin{aligned}\psi(z + L_1 + iL_2) &= \psi(z + L_1)\exp\{-iL_2\,(z + L_1) + \tfrac{1}{2}L_2^2 + \delta_2\}\\ &= \psi(z)\exp\{(L_1 - iL_2)\,z + \tfrac{1}{2}|L_1 + iL_2|^2 + i\delta_1 + i\delta_2 - iL_1L_2\}\\ &= \psi(z + iL_2)\exp\{L_1(z + iL_2) + \tfrac{1}{2}L_1^2 + \delta_1\}\\ &= \psi(z)\exp\{(L_1 - iL_2)\,z + \tfrac{1}{2}|L_1 + iL_2|^2 + i\delta_1 + i\delta_2 + iL_1L_2\}\end{aligned} \tag{4}$$

Hence

$$2L_1L_2 = 2N\pi$$

which is the Dirac–Weil–Kostant–Souriau quantization condition. Let us conclude this section by giving the explicit expression for $\langle\psi|H|\psi\rangle$, which exhibits H as a positive operator:

$$\langle\psi|H|\psi\rangle = \int_{\mathbb{T}^2} [dz]\; e^{-|z|^2}|\bar{\partial}\psi|^2$$

This is a general result for quantum mechanics on Kaehler manifolds [7], from which we get the general result that the ground state coincides with the subspace of holomorphic sections ($\bar{\partial}\psi = 0$).

5. FINITE-DIMENSIONAL LANDAU LEVELS

We can now compute the solutions of Schrödinger equation belonging to the ground state. These are given by holomorphic functions satisfying the boundary conditions (3). Let us choose $\delta_1 = \delta_2 = 0$. Setting $\psi(z) = \exp\{\frac{1}{2}z^2\}$ $\theta(z)$, we find that θ must be periodic with real period L_1, hence it can be expanded in a Fourier series $\theta(z) = \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n z/L_1}$. It follows that

$$\begin{aligned}\psi(z + iL_2) &= e^{(1/2)(z+iL_2)^2} \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n z/L_1}\, e^{-2\pi n L_2/L_1}\\ &= e^{(1/2)z^2} \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n z/L_1}\, e^{-iL_2 z+(1/2)L_2^2}\end{aligned}$$

which gives

$$e^{2iL_2 z - L_2^2} \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n z/L_1}\, e^{-2n\pi L_2/L_1} = \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n z/L_1}$$

Making use of Dirac's quantization ($L_2 = N\pi/L_1$), we get the condition

$$\sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i (n+N) z/L_1}\, e^{-2n\pi L_2/L_1 - L_2^2} = \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i n z/L_1}$$

which is readily transformed into the recurrence relation

$$c_n = c_{n-N}\, e^{-2n\pi L_2/L_1 + 2N\pi L_2/L_1 - L_2^2}$$

whose solution is

$$c_n = e^{-\pi n^2 L_2/(L_1 N)}\, b_n$$

where b_n is such that $b_n = b_{n+N}$. Hence *there are N orthogonal solutions* given by

$$\left\{\psi_\nu(z) = \mathcal{N}_\nu\, e^{(1/2)z^2} \sum_{n \equiv \nu \bmod (N)} \exp\left(-\frac{\pi n^2 L_2}{N L_1} + \frac{2n\pi i z}{L_1}\right) \middle|\, \nu = 0, 1, \ldots, N-1\right\}$$

We can obtain a new representation in terms Gaussian functions, very convenient for a practical calculation of ψ, by applying the Poisson summation formula (see, e.g., ref. 8). We find

$$\psi_\nu(z) = \mathcal{N}_\nu\, e^{(1/2)z^2 + 2\pi i \nu z/L_1} \sum_{n=-\infty}^{\infty} \exp\{-\,(z + nL_1/N + i\nu\, L_2/N)^2\}$$

Higher levels can be simply obtained by applying the covariant creation operator $\partial - \bar{z}$ to each ψ_ν.

6. TRANSLATION SYMMETRY BREAKING

The main question which started this investigation was the following: What happens to the translation symmetry of the torus? The question is motivated by the fact that unitary translations are realized as projective representations with a “central charge” given by the magnetic field strength. Hence they cannot live in a finite-dimensional space (see Appendix A). Before going on to analyze the problem in great detail, just observe that under the assumption that such a translation symmetry would nevertheless survive in some way, we should see it as a property of the ground state, i.e., there must exist a finite unitary matrix $t_{\mu\nu}$ such that $(T_a\,\psi_\nu)(z) = \Sigma_\mu\, t_{\nu\mu}(a)\psi_\mu(z)$. It would follow that the density matrix $\rho_N(z) = \Sigma_\nu\,|\psi_\nu(z)\,|^2$ should then be translation invariant, i.e., constant on the torus. If we calculate ρ_N for the first few values of N, we immediately find that this is not so. The density ρ exhibits a series of regularly spaced bumps, precisely at the location $(n_1L_1 + n_2L_2)/N$ (see Figs. 1 and 2, where the deviation from uniformity is plotted for the first two Landau levels at various values of the magnetic charge).

As is clear from the figures, translation symmetry is broken, presumably to $Z_N \times Z_N$, but the breaking tends to be weaker at high N [a variation of $O(10^{-N})$]. Is there a simple explanation of this symmetry breaking? The point is that we can easily implement compact translations in the same way as we can do in the noncompact case. The unitary operators are given by

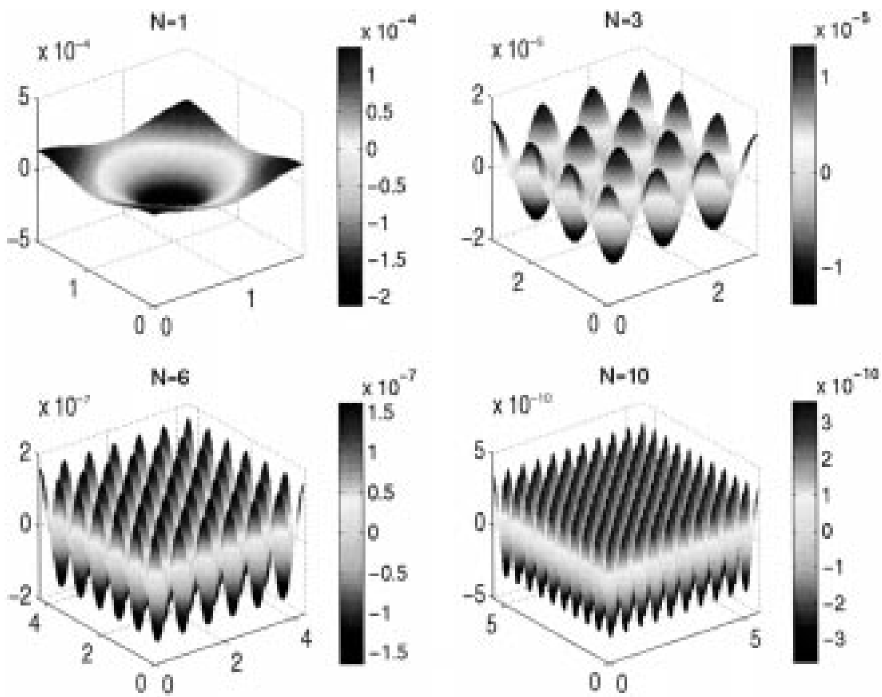

Fig. 1. Deviation from uniformity of $\rho = \Sigma_{\nu=0}^{N-1},\ |\psi_\nu(z)|^2$, $N = 1, 3, 6, 10$.

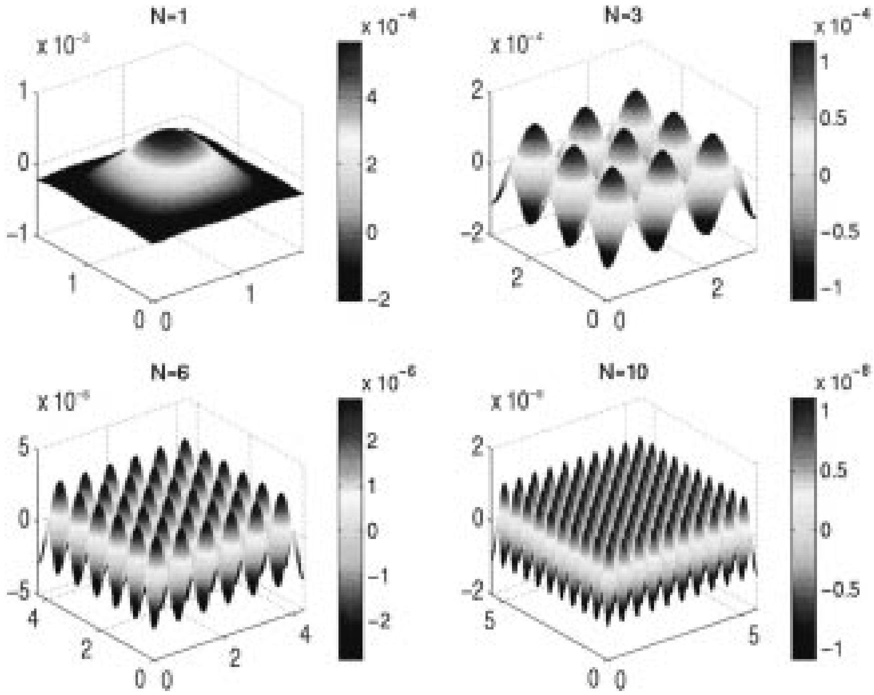

Fig. 2. Deviation from uniformity of ρ in the second Landau level, $N = 1, 3, 6, 10$.

$$(T_a\psi)(z) = e^{\bar{a}z-(1/2)|a|^2}\,\psi(z - a)$$

where the value of ψ should be found through the twisted periodicity conditions given in Eq. (3). It is readily checked that:

1. T_a *formally* commute with the Hamiltonian, i.e., with the differential operator of Eq. (2).
2. $T_aT_bT_{-a}T_{-b} = \exp\{\bar{a}b - a\bar{b}\}$
3. T_a does *not* in general leave the ground state invariant, i.e., invariance is maintained only if Na is trivial, that is, $a = (n_1L_1 + in_2L_2)/N$.
4. The formal infinitesimal generators of T_a, namely $\mathfrak{p}_1 = iz - i(\partial + \bar{\partial})$ and $\mathfrak{p}_2 = -iz - i(\partial - \bar{\partial})$, *do not* leave the space of sections [Eq. (3)] invariant.

To begin with the last statement, it is clear that we may consider the linear combinations ∂ and $z - \bar{\partial}$, neither of which is such as to transform sections into sections. From the group point of view, let l be a translation in $\mathbb{Z}_2$, i.e., $l = k_1L_1 + ik_2L_2$, $k \in \mathbb{Z}$. Let us consider $T_a\psi$.

We find

$$\begin{aligned}(T_a\psi)(z + l) &= \exp\{\bar{a}(z + l) - \tfrac{1}{2}|a|^2\} \exp\{\bar{l}(z - a) + \tfrac{1}{2}|l|^2\}\psi(z - a) \\ &= (T_a\psi)(2) \exp\{\bar{l}z + \tfrac{1}{2}|l|^2 + \bar{a}l - a\bar{l}\}\end{aligned}$$

We conclude that a translated section satisfies boundary conditions with a different choice of the constants δ_1, δ_2, hence the bundle structure is not invariant under translation, except for

$$\bar{a}l - a\bar{l} = 2i\Im\{\bar{a}l\} \in 2\pi i\mathbb{Z}$$

which occurs precisely when $a = (n_1L_1 + in_2L_2)/N$ [$\Im\{\bar{a}l\} = (n_1k_2 - n_2k_1)L_1L_2/N = (n_1k_2 - n_2k_1)\pi$, by Dirac's quantization].

7. CONCLUSIONS

The problem of a constant magnetic field transverse to a torus raises the problem of translational symmetry. By quantizing the system according to the standard mathematical formulation of gauge theory, we have shown that the symmetry is broken to $Z_N \times Z_N$. The conclusion to which one is led by this result is that the ambiguity in quantization, namely the two arbitrary phases δ_1, δ_2, entering in the definition of the domain of the Hamiltonian operator, represent some physical degree of freedom of the magnetic charge distribution generating the uniform field on the torus: monopole charges have, so to speak, horns. The effect is purely quantum mechanical and we empirically established that it vanishes approximately as $\exp\{-O(B/\hbar)\}$. The mathematical roots of the result are the classic theorems of Weil (see ref. 12, Chapter VI, Proposition 3, n. 3); a thorough study of θ-functions can be found in ref. 3.

APPENDIX A. A GROUP-THEORETIC WINTNER THEOREM

Wintner's theorem (see ref. 10) states that the identity operator in a Hilbert space cannot be the commutator of two bounded operators. There is a poor's man version of the theorem. Let $U(a)$ and $V(b)$ be unitary operators satisfying the canonical commutation relations (at the group level)

$$U(a)V(b)U(-a)V(-b) = e^{\bar{a}b - a\bar{b}} \qquad (a, b \in \mathbb{C})$$

Then U and V cannot be finite-dimensional matrices.

Proof. Just evaluate the determinant of both sides to get

$$1 = \exp\{2iN\Im(\bar{a}b)\} \qquad \text{with} \quad N = \dim(U)$$

This is a contradiction, since the r.h.s. can assume any value on the unit circle. This last equation shows that we may take a and b in a finite subgroup and preserve the commutation relation: let $Z_N = \{(n_1L_1 + in_2L_2)/N | n_i \in \mathbb{Z}\}$; then the condition is satisfied precisely if $L_1L_2 = N\pi$.

APPENDIX B. DIRAC–WEIL–KOSTANT–SOURIAU QUANTIZATION CONDITION

A general theorem (ref. 6, Theorem 21.1) relates the dimension of spaces of closed holomorphic forms on complex vector bundles to geometrical objects, namely Chern and Todd classes of the base space and of the bundle. In the simple case of a line bundle (fiber equal to $\mathbb{C}$) over a complex two-dimensional Riemann surface, the theorem reduces to a simple result which has a very intuitive flavor from the point of view of geometric quantization: the dimension of the physical Hilbert space coincides with the volume of phase space in units $\hbar$ plus a constant given by half the Euler characteristic of the surface. This in turn implies that the volume of phase space must be an integer. We report here what appears to be the simplest proof, covering Dirac's quantization condition, combining ideas from refs. 1 and 11. Let us build a triangulation of the surface $\mathcal{M}$ with vertices $\alpha, \beta, \gamma, \ldots$. Let $\mathcal{U}_\alpha$ denote the union of all triangles having α as vertex. By taking a sufficiently fine mesh, nonempty intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ consist of the union of two triangles which share the side $\alpha - \beta$. A gauge field on $\mathcal{M}$ is given by a closed two-form $\mathbf{B}$; in each local chart $\mathcal{U}_\alpha$, we define a potential $\mathbf{A}_\alpha$ such that $\mathbf{B} = d\mathbf{A}_\alpha$ in $\mathcal{U}_\alpha$. According to the Poincaré lemma, for neighboring local charts, we have

$$\mathbf{A}_\alpha - \mathbf{A}_\beta = d\chi_{\alpha\beta}$$

with differentiable *transition functions* $\chi_{\alpha\beta}$ which are antisymmetric in their indices. On triple intersections (any triangle $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$), we have

$$\mathbf{A}_\alpha - \mathbf{A}_\beta = d\chi_{\alpha\beta}, \qquad \mathbf{A}_\beta - \mathbf{A}_\gamma = d\chi_{\beta\gamma}, \qquad \mathbf{A}_\gamma - \mathbf{A}_\alpha = d\chi_{\gamma\alpha}$$

It follows that $c_{\alpha\beta\gamma} \equiv \chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha}$ is constant on each triangle.[4] Let us introduce a line bundle associated to $\mathbf{B}$: it is given locally by a direct product $\mathcal{U}_\alpha \times \mathbb{C}$ in such a way that in any overlap the complex fibers are connected by[5]

$$\zeta_\alpha = \zeta_\beta \exp\{i\chi_{\alpha\beta}\}$$

For consistency, on any triple overlap it must hold that

$$\exp\{i\chi_{\alpha\beta} + i\chi_{\beta\gamma} + i\chi_{\gamma\alpha}\} = 1$$

which implies thar $c_{\alpha\beta\gamma}$ must be an integer multiple of 2π. This is usually

[4] It is useful to regard the relation between A, χ, and c in terms of the coboundary operator: $(\delta A)_{\alpha\beta} = d\chi_{\alpha\beta}$, $c_{\alpha\beta\gamma} = (\delta\chi)_{\alpha\beta\gamma}$. See ref. 1.

[5] In the physical application, the phase is $e\chi/\hbar c$.

referred as the Weil theorem on holomorphic vector bundles (ref. 12, Chapter V, Proposition 1, n. 4).

The key result for our purposes is the following:

Theorem. The integral $\int_{\mathcal{M}} \mathbf{B}$ coincides with the discrete sum $\Sigma_\Delta c_\Delta$, where Δ runs over all triangles of the mesh.

Proof. The following, purely algebraic identity holds (ref. 11, p. 131):

$$\begin{aligned}
\int_{\Delta_{\alpha\beta\gamma}} \mathbf{B} &= \frac{1}{3} \oint_{\partial\Delta_{\alpha\beta\gamma}} (\mathbf{A}_\alpha + \mathbf{A}_\beta + \mathbf{A}_\gamma) \\
&= \frac{1}{3} [(\chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha})(\alpha) + (\chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha})(\beta) \\
&\quad + (\chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha})(\gamma)] - \frac{1}{2} \{(\chi_{\alpha\beta}(\alpha) + \chi_{\alpha\beta}(\beta)) \\
&\quad + (\chi_{\beta\gamma}(\beta) + \chi_{\beta\gamma}(\gamma)) + (\chi_{\gamma\alpha}(\gamma) + \chi_{\gamma\alpha}(\alpha))\} \\
&\quad + \frac{1}{2} \left\{ \int_{\alpha\beta} (\mathbf{A}_\alpha + \mathbf{A}_\beta) + \int_{\beta\gamma} (\mathbf{A}_\beta + \mathbf{A}_\gamma) + \int_{\gamma\alpha} (\mathbf{A}_\gamma + \mathbf{A}_\alpha) \right\}
\end{aligned}$$

Notice that the terms in curly brackets average to zero when we sum over the whole triangulation, while the terms in square brackets are precisely the cocycle c_Δ, whose value is constant on the triangle. Hence we get

$$\int_{\mathcal{M}} \mathbf{B} = \sum_\Delta c_\Delta$$

and as a result, the flux of **B** is quantized.

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